Geometric Phase and Quantum Phase Transition in the Lipkin-Meshkov-Glick model

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The relation between the geometric phase and quantum phase transition has been discussed in the Lipkin-Meshkov-Glick model. Our calculation shows the ability of geometric phase of the ground state to mark quantum phase transition in this model. The possibility of the geometric phase or its derivatives as the universal order parameter of characterizing quantum phase transitions has been also discussed.

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I. INTRODUCTION

Recently the understanding of quantum phase transition [1] has emerged from the fundamentals of quantum mechanics, especially from the entanglement point of view [2]. The ground state structure have been shown to be affected appreciably by the critical points, as illustrated initially in the one-dimensional Ising model with transverse magnetic field [3]. Intriguing by these pioneering works, a great deal of efforts have attributed to this region [4, 5, 6]. It originates from the belief that quantum phase transition should be connected to the entanglement or its derivatives in many-body systems [2]. However the correspondence between the entanglement and quantum phase transition in many-body systems is ambiguous [7, 8], that is because of the absence of the proper measurement of entanglement in many-body systems. On the other hand, geometric phase [9] as a measurement of the curvature of the Hilbert space, has been first connected with the quantum phase transition in the one-dimension spin-chain system [10], in which the topological character of the relative geometric phase between the first excited state and the ground state have been shown the ability of detecting the critical points. Furthermore, Zhu extended this study under the thermodynamic limit and found that the geometric phase of the ground state in the one-dimensional XY model was non-analytical and its derivative with the coupling constant was divergent closed to critical points [11]. Moreover the scaling behavior of the geometric phase of ground state clearly distinguished two different types of quantum phase transitions at the critical point in the one-dimensional XY model. Consequently the general theory about the relation between geometric phase of the ground state and the critical point has been constructed [12], in which the topological property of Berry's loops including critical points in the parameter space has been shown the ability of detecting the critical points.

Although the theory is rapidly developing, the verifications of examples only focus on the one-dimension XY spin-chain systems. The reason is that XY model, which have been shown first-order phase transition closed to the critical point, can be converted into spinless fermionic system by Jordan-Wigner transformation, and the energy spectrum can be determined exactly [13]. However the interacting between spins in this model is short-range (the neighbor-nearest coupling) and the anisotropy of Heisenberg interaction is indispensable for the construction of long-range order [13]. Recently the Berry phase in Dick model has been examined under thermodynamics limit [14]. The authors showed that the Berry phase displayed the non-analyticity close to critical point where this model exhibits a second-order phase transition, and its derivative with the coupling constant was also discontinued and showed a cusp closed to this point. The discontinuity of Berry phase is very similar to that in XY model, but we should point out that the two model belong to different orders of quantum phase transitions respectively. In this paper, we will show a special situation, in which the geometric phase of the ground state behaves differently and itself distinguishes the different quantum phase transitions.

In particular we study the geometric phase in a system of spins with a collective coupling described by the Lipkin-Meshkov-Glick (LMG) model [15], which first introduced forty years ago in nuclear physics. This model has received much attention because of his apparent simplicity and popularity in the past. It provides a simple description of the tunneling of bosons between two degenerate levels and can thus be used to describe many physical systems such as two-mode Bose-Einstein condensates [16] or Josephson junctions [17]. More recently the entanglement in this model has received great attention because of available numerical calculations and plentiful phase diagram [8, 18]. Under the thermodynamics limit, its phase diagram can be simply established by a semiclassical approach [19]. For large particle number N but finite, the situation is complicated and the numerical analysis was implemented using the continuous unitary transformations [20]. The significant difference between this model and XY model is the long-range interaction, and the system cannot be converted into the spinless fermionic system. Hence it is of great interest to study the relation of geometric phase and phase transition in this model. As will display in the remaining of this paper, the geometric phase of ground state in this model

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behaves differently and reflects faithfully the existence of the critical points.

The paper is organized as follows. In Sec. II, we first introduce the LMG model, and in the limit of large N but finite, we obtain the ground state analytically by the Holstein-Primakoff representation and calculate the geometric phase. In order to show the university of our results, in Sec. III, we study the phase transition in a more complex situation. It is surprising that the geometric phase of ground state detects very rigorously the critical points. Finally we discuss the implications of our results and the differences from the XY model.

II. THE LIPKIN-MESHKOV-GLICK MODEL: BIAXIAL CASE

The LMG model describes a set of N spins half coupled to all others with a strength independent of the position and the nature of the elements and a magnetic field in the z direction. The Hamiltonian can be written

$$H = -\frac{1}{N}(S_x^2 + \gamma S_y^2) - hS_z, \tag{1}$$

in which $S_\alpha = \sum_{i=1}^N \sigma_\alpha^i/2(\alpha = x, y, z)$ and the σ_α is the Pauli operator, N is the total particle number in this system. The prefactor $1/N$ is essential to ensure the convergence of the free energy per spin in the thermodynamic limit. For any anisotropy parameter $\gamma \in [0, 1]$, the Hamiltonian (1) preserves the total spin and does not couple the state having spin pointing in the direction perpendicular to the field, namely

$$[H, \mathbf{S}^2] = 0, [H, \prod_{i=1}^N \sigma_z^i] = 0. \tag{2}$$

An important character that differentiates LMG model from the XY model is the long-rang interaction between particles, which induces a second-order phase transition at $h = 1$ when $N \to \infty$ [19].

The diagonalization of Eq. (1) can be obtained by introducing the Holstein-Primakoff representation of the spin operator and then truncate the resulting bosonic Hamiltonian to lowest order [20]. Consequently we diagonalize it thanks to the Bogoliubov transformation. The first thing is to perform a rotation of the spin operators around the y direction, that makes the z axis along the so-called semiclassical magnetization [20] in which the Hamiltonian Eq. (1) has the minimal value in the semiclassical approximation. This can be done as

$$\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \tilde{S}_x \\ \tilde{S}_y \\ \tilde{S}_z \end{pmatrix} \tag{3}$$

in which $\theta = 0$ for $h > 1$, $\theta = \arccos h$ for $0 < h < 1$. Then the Hamiltonian Eq. (1) is converted into

$$\begin{aligned} H &= -\frac{1}{N}[\sin^2\theta \tilde{S}_z{}^2 + \frac{\cos^2\theta + \gamma}{2}(\tilde{S}^2 - \tilde{S}_z{}^2)] \\ &+ \frac{\sin 2\theta}{N}(\tilde{S}^+\tilde{S}_z + \tilde{S}^-\tilde{S}_z + h.c.) - \frac{\cos^2\theta - \gamma}{4N}(\tilde{S}^{+2} + \tilde{S}^{-2}) \\ &- h_z\cos\theta\tilde{S}_z - \frac{h_z}{2}\sin\theta(\tilde{S}^+ + \tilde{S}^-), \end{aligned} \tag{4}$$

in which $\tilde{S}^\pm = \tilde{S}_x \pm i\tilde{S}_y$.

In order to obtain the geometric phase of ground state, we consider the system has a rotation $\tilde{g}(\phi)$ around the new z direction. The Hamiltonian becomes

$$H(\phi) = \tilde{g}(\phi)H\tilde{g}^\dagger(\phi). \tag{5}$$

in which $\tilde{g}(\phi) = e^{i\phi\tilde{S}_z}$. Then we can use the Holstein-Primakoff representation,

$$\begin{aligned} \tilde{S}_z(\phi) &= N/2 - a^\dagger a, \\ \tilde{S}^+(\phi) &= (N - a^\dagger a)^{1/2} a e^{i\phi}, \\ \tilde{S}^-(\phi) &= a^\dagger e^{-i\phi}(N - a^\dagger a)^{1/2} \end{aligned} \tag{6}$$

in which $a^{(\dagger)}$ is bosonic operator. Since the z axis is along the semiclassical magnetization, $a^\dagger a/N \ll 1$ is a reasonable assumption under low-energy approximation, in which N is large but finite. Keeping the terms of order $N, N^{1/2}, N^0$, Eq. (5) becomes

$$H(\phi) = Ne + \Delta a^\dagger a + \Gamma(a^{\dagger 2}e^{-2i\phi} + a^2 e^{2i\phi}), \tag{7}$$

in which

$$\begin{aligned} e &= -\frac{1}{4}(\sin^2\theta + 2h\cos\theta), \\ \Delta &= \sin^2\theta - \frac{\gamma + \cos^2\theta}{2} + h\cos\theta \\ \Gamma &= \frac{\gamma - \cos^2\theta}{4} \end{aligned} \tag{8}$$

Obviously the equation above can be diagonalized by the standard Bogoliubov transformation,

$$b(\phi) = \cosh x a e^{i\phi} + \sinh x a^\dagger e^{-i\phi} \tag{9}$$

Then the Hamiltonian is

$$H_{diag}(\phi) = Ne + \sigma + \Delta^D b^\dagger(\phi) b(\phi), \tag{10}$$

in which,

$$\begin{aligned} \sigma &= \frac{\Delta}{2}(\sqrt{1 - \epsilon^2} - 1) \\ \Delta^D &= \Delta\sqrt{1 - \epsilon^2} \\ \epsilon &= \frac{2\Gamma}{\Delta} = \tanh 2x = \begin{cases} -\frac{1-\gamma}{2h-1-\gamma}, & h > 1 \\ -\frac{h^2-\gamma}{2-h^2-\gamma}, & 0 < h < 1 \end{cases} \end{aligned} \tag{11}$$

It should be careful about the physical interpretation of Δ^D, which may not describe true gap of the system [20].

Now it is time to determine the ground state $|g(\phi)\rangle$, which can be obtained by applying the relation,

$$b(\phi)|g(\phi)\rangle = 0 \tag{12}$$

Substituting Eq. (9) into the equation above, one obtains the ground state,

$$|g(\phi)\rangle = \frac{1}{C}\sum_{n=0}^{[N/2]}\sqrt{\frac{(2n-1)!!}{2n!!}}(-\frac{e^{-i\phi}\sinh x}{e^{i\phi}\cosh x})^{n-1}(-\sqrt{2}e^{-i\phi}\sinh x)|2n\rangle, \tag{13}$$

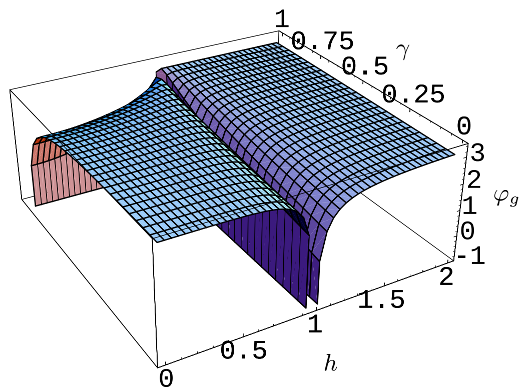

FIG. 1: The geometric phase φ_g [Arc] vs. the anisotropic parameter γ and h. For specification, we have chosen the summation from 0 to 100 in the expression of φ_g (Eq. (14)). The divergent character of φ_g is clearly displayed at $h \to 1$ in this figure.

in which $n!! = n(n-2)(n-4)\cdots$ and $n!! = 1$ for $n \leq 0$. $|n\rangle$ is the Fock state of bosonic operator $a^{(\dagger)}$ and the normalized constant is $C^2 = \sum_{n=0}^{[N/2]} 2\sinh^2 x \frac{(2n-1)!!}{2n!!}\tanh^{2(n-1)} x$. One should note that in order that the summation is convergent, $|\tanh x| \leq 1$.

The geometric phase of the ground state, accumulated by changing ϕ from 0 to π, is determined by $\varphi_g = -i\int_0^\pi d\phi \langle g(\phi)|\partial_\phi|g(\phi)\rangle$. The direct calculation shows

$$\varphi_g = \pi[1 - \frac{\sum_{n=0}^{[N/2]} 2n\frac{(2n-1)!!}{2n!!}\tanh^{2(n-1)} x}{\sum_{n=0}^{[N/2]}\frac{(2n-1)!!}{2n!!}\tanh^{2(n-1)} x}] \tag{14}$$

which have been plotted with γ and h in Fig. 1. It is obvious that φ_g is divergent at the point $h = 1$ where the LMG model has been proved to experience a second-order phase transition, independent of the anisotropy γ[19]. This divergency has never been explored in the previous studies [10, 11, 12, 14] and shows distinguished character from the XY and Dicke models. In fact, since the adiabatic condition has been destroyed when $h \to 1$ because of the degeneracy between the ground state and the excited state and the degeneracy enhances geometric phase [9], this phenomenon is natural. However, the essential reason is the collective interaction in the LMG model, which is absent in the XY model and makes the long-range correlations in this system.

The finite size effect is also examined in this case by choosing different N, which can be shown in the Fig. 2(a) and (b). In this figure, we only draw for $N = 4, 1000$

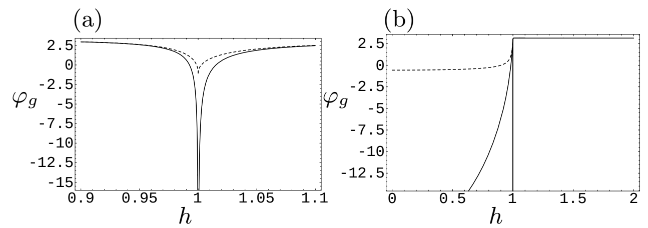

FIG. 2: φ_g [Arc] vs. h with different particle number N. We have chosen $\gamma = 0.5$ (a) and $\gamma = 1$ (b) for this plot. The dashed and solid lines correspond respectively to $N = 4, 1000$.

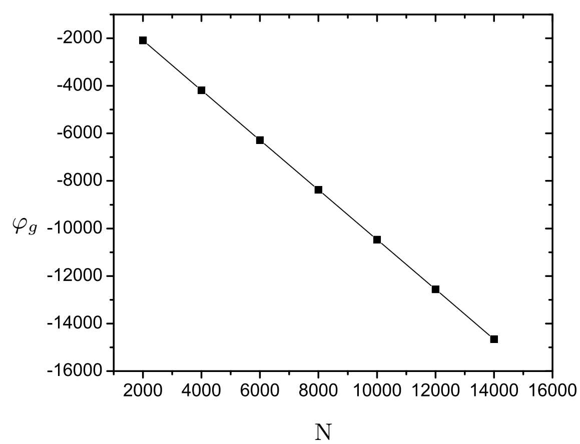

FIG. 3: The scaling behavior of φ_g [Arc] vs. N with $\gamma = 0.5$ when $h \to 1$. The slope of the line is close to -1.

and for higher values of N, the curves almost do not change. From the figures we note that the divergency of φ_g seems to be insensitive to N. However, since phase transition only happens under thermodynamic limit, this phenomenon attributes to the ground-state degeneracy at the critical point.

The scaling behavior of φ_g plotted with N has also been explored in Fig. 3. It is obvious that one can obtain

$$\varphi_g \approx -N. \tag{15}$$

Furthermore the scaling is independent of γ, which means that for different γ, the phase transitions belong to the same university class. We also check the scaling behavior when $\gamma = 1$ and find that this case has the same behavior as displayed in Fig. 3. It means that the phase transition for $\gamma = 1$ is the same university class as that for $\gamma \neq 1$. This phenomenon is different from the XY model,

in which the isotropic and anisotropic interactions belong respectively to different university classes [11], that comes from the collective interaction in the LMG model.

III. UNIAXIAL MODEL

In this part, we will discuss the phase transition in a more complex system, which is the generalization of the LMG model Eq. (1). The Hamiltonian can be written as

$$H = -\frac{1}{N}S_x^2 - h_x S_x - h_z S_z, \tag{16}$$

with $h_z > 0$. The phase diagram of this model is obviously dependent on the both parameters h_x, h_z, and moreover a proper order parameter characterizing the transition is difficult to build. Recently the correspondence between this model and a two-level boson problem introduced in nuclear physics has been constructed, which permits one to get an order parameter [21]. The phase transition is then clear in this model; a first-order transition occurs at $h_x = 0$ with $h_z < 1$ and a second-order one occurs at $h_x = 0$ with $h_z = 1$. For $h_z > 1$ or $h_x \neq 0$, no transition is found.

Now we will try to determine the phase transition by calculating the geometric phase of the ground state. Similar procedures as in previous section can be applied for this purpose. The first step is to make the system have a rotation around the z direction and then the Hamiltonian 16 becomes $H(\phi) = g(\phi)Hg^\dagger(\phi)$. Next step is to introduce the Holstein-Primakoff transformation Eq. (6). One should note that the approximation $a^\dagger a/N \ll 1$ is invalid in this case since we cannot find the semiclassical magnetization. However a simple canonical transformation can be used to resolve this problem, $a^{(\dagger)}e^{(-)i\phi} = b^{(\dagger)}e^{(-)i\phi} + \sqrt{N}\lambda$ with $|\lambda| < 1$. This transformation provides a macroscopic expectation value of S_z which is order of N, and then one has $b^\dagger b/N \ll 1$. Under the limit that N is large but finite, it is enough to expand $H(\phi)$ to the order N^0. After the exhausted calculation, $H(\phi)$ is written as,

$$\begin{aligned} H(\phi) = \quad & e_0(\lambda) + \Omega(\lambda)(be^{i\phi} + b^\dagger e^{-i\phi}) + \\ & \Gamma(\lambda)(b^2 e^{2i\phi} + b^{\dagger 2} e^{-2i\phi}) + \Delta(\lambda) b^\dagger b, \end{aligned} \tag{17}$$

in which,

$$\begin{aligned} e_0(\lambda) &= -N[\frac{h_z}{2}(1-2\lambda^2) + \lambda^2(1-\lambda^2) + h_x(1-\lambda^2)] \\ &- (\frac{1}{4} - \lambda^2) - h_x \frac{\lambda(2-\lambda^2)}{8(1-\lambda^2)^{3/2}} \\ \Omega(\lambda) &= \sqrt{N}[\lambda h_z - \frac{h_x(1-2\lambda^2)}{2\sqrt{1-\lambda^2}} - \lambda(1-2\lambda^2)] \\ \Gamma(\lambda) &= -\frac{1-5\lambda^2}{4} + h_x \frac{\lambda(2-\lambda^2)}{8(1-\lambda^2)^{3/2}} \\ \Delta(\lambda) &= h_z - \frac{1-7\lambda^2}{2} + h_x \frac{\lambda(4-3\lambda^2)}{4(1-\lambda^2)^{3/2}}. \end{aligned} \tag{18}$$

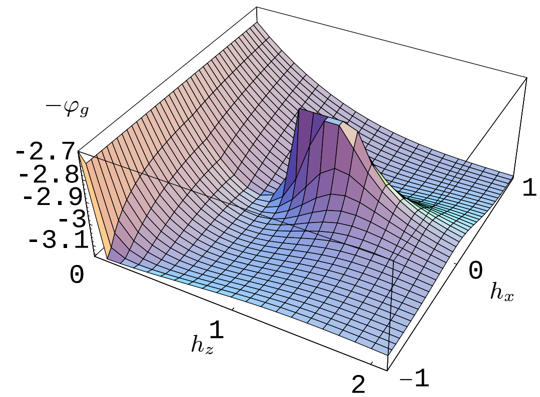

FIG. 4: The geometric phase φ_g [Arc] vs. h_x and h_z. We have chosen the summation from 0 to 100 in the expression of φ_g Eq. (14), and for the convenience of viewport, we have drawn for $-\varphi_g$ in this plot.

The crucial step is to choose λ_0 properly in order that the linear term (the second term in Eq. (17)) is vanishing. It can be realized by solving the following equation,

$$\lambda_0 h_z - \frac{h_x(1-2\lambda_0^2)}{2\sqrt{1-\lambda_0^2}} - \lambda_0(1-2\lambda_0^2) = 0. \tag{19}$$

In fact the equation above can be reduced into the biquadratic equation $(h_z - y)^2(1-y^2) - h_x^2 y^2 = 0$ with $y = 1 - 2\lambda_0^2$, which can be solved numerically.

Substitute λ_0 into Eq. (17) and then one get the quadratic Hamiltonian,

$$H = e_0(\lambda_0) + \Gamma(\lambda_0)(b^2 e^{2i\phi} + b^{\dagger 2} e^{-2i\phi}) + \Delta(\lambda_0) b^\dagger b, \tag{20}$$

which obviously could be diagonalized by the standard Bogoliubov transformation. Consequently the geometric phase of ground state can be determined directly, which has the same form as Eq. (14), but different definition of x, determined by the equation $\tanh 2x = \frac{2\Gamma(\lambda_0)}{\Delta(\lambda_0)}$.

A schematic demonstration of the geometric phase is presented in Fig. 4. It is obvious that there are two regions divided by $h_z = 1$, and the geometric phase is divergent at point $h_x = 0, h_z = 1$. In the region $h_z < 1$, geometric phase is non-analytical at point $h_x = 0$, which means the appearance of phase transition, and in the other region, the geometric phase is the smooth function of h_z, h_x and no phase transition is found from the figure. This phenomenon is consistent with the conclusion in Ref. [8], but in our calculation we do not need to find a proper order parameter to characterize the phase transition and the geometric phase of ground state faithfully marks these transitions. A detailed demonstration is also provided in Fig. 5. From the figures, one easily finds that the geometric phase φ_g has a cusp at $h_x = 0$ for $h_z < 1$ (see Fig. 5(a)), which implies the first-order phase transition. Furthermore φ_g is divergent at $h_x = 0$ when $h_z = 1$ (see Fig. 5(b)), which is similar to that in the standard LMG model (see Figs. 1and 2) and means

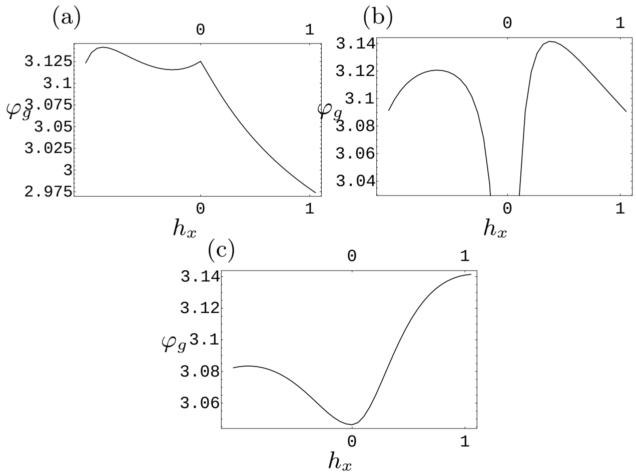

FIG. 5: The geometric phase φ_g [Arc] vs. h_x with different h_z. We have chosen $h_z = 0.5$ (a), $h_z = 1$ (b) and $h_z = 2$ (c) for this plot. The figure (b) has been compressed because of the divergency of the value φ.

that there is a second-order transition. For $h_z > 1$ φ_g is the smooth function of h_x, h_z and there is no transition (see Fig. 5(c)).

It is surprising that the geometric phase of ground state itself can differentiate phase transition without need of introducing a proper order parameter, which may be difficult to find. Thus it is a natural speculation that the geometric phase of ground state can serve as a universal order parameter. Further discussions will be presented in the next part.

IV. CONCLUSIONS AND DISCUSSIONS

The geometric phases of ground state in the LMG model and its generalization have been discussed in this paper. Our calculations show that the geometric phase faithfully reflects the phase transition in this model; when there is a second-order phase transition, the geometric phase behaves divergent (see Figs. 1 and 5(b)). However when there is a first-order transition, non-analyticity of geometric phase appears at the critical point (see Fig. 5(a)). These phenomena may come from the degeneracy of the ground state in the system, which usually induces the mixture of different phases. The different behaviors of geometric phase closed to critical points could originate from the different excitations (i.e. gapped or gapless). Furthermore we find that the energy of ground state is not a good parameter of marking the phase transition, because the energy of ground state is degenerate independent of the phase transition is second-order or first-order. Recently Tian and Lin have shown that the continuous quantum phase transition are actually caused by level crossing of the low-lying excited states of the system [22]. This conclusion also shows that the only energy of the ground state is not suitable for the characterization of quantum phase transition.

This leads to a question what physical quantity is suitable for characterizing the quantum phase transition. With respect of the work Ref [11] and our calculations, it seems to provide us an hint that the geometric phase of ground state or its derivatives could serve as an universal order parameter to characterize different phase transitions. This speculation is natural since the geometric phase faithfully measure the curvature of the Hilbert space (or phase space for classical mechanics) and the broken of symmetry of Hilbert space must be reflected in the geometric phase.

Another aspect of importance is the differences between our model and the one-dimensional XY model. The crucial point is the collective interaction in LMG model, which is absent in the XY model. A main result of this interaction is that the LMG model cannot be converted into the spinless fermion system. This fact makes the conclusion different from that in Ref. [12], in which the topological behavior of Berry phase can detect the critical point. However, in this model, the geometric phase is divergent when there is second-order transition and the detection maybe invalid.

In conclusion the relation between the geometric phase of ground state and the phase transition in LMG model has been constructed in this paper. Different from the results in the one-dimensional XY model, the singularity of the geometric phase itself can serve as the signature of critical points. Moreover we discuss the possibility of the geometric phase of ground state or its derivatives serving as the universal order parameter to characterize the quantum phase transitions.

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